Orthogonal Asymptotic Lines on Surfaces Immersed in $\mathbf{R}^4$

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Abstract

In this paper we study some properties of surfaces immersed in $\mathbf{R}^4$ whose asymptotic lines are orthogonal. We also analyze necessary and sufficient conditions for the hypersphericity of surfaces in $\mathbf{R}^4$.

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1 Introduction

There are two different ways to construct line fields on surfaces immersed in $\mathbf{R}^4$. The first one consists in considering the ellipse of curvature in the normal bundle of the surface and taking the pull back of points on this ellipse to define tangent direction fields. Examples of this approach are given by: the *lines of axial curvature*, along which the second fundamental form points in the direction of the large and the small axes of the ellipse of curvature; the *mean directionally curved lines*, along which the second fundamental form points in the direction of the mean curvature vector; and the *asymptotic lines*, along which the second fundamental form points in the direction of the tangent lines to the ellipse of curvature.

The other way consists in defining the *ν-principal curvature lines*, along which the surface bends extremally in the direction of the normal vector ν. To this end, we need to take an unitary normal vector field ν and follow the classical approach for surfaces immersed in $\mathbf{R}^3$.

The lines of axial curvature are globally defined and their singularities are the axiumbilic points where the ellipse of curvature becomes either a circle or a point. The axiumbilic points and the lines of axial curvature are assembled into two axial configurations. The first one is defined by the axiumbilics and the field of orthogonal lines on which the surface is curved along the large axis of the ellipse of curvature. The second one is defined by the axiumbilics and the field of orthogonal lines on which the surface is curved along the small axis of the ellipse of curvature. Each axial configuration is a net consisting of orthogonal curves and axiumbilic points. Therefore a line of axial curvature is not necessarily a simple regular curve; it can be immersed with transversal crossings. The differential equation of lines of axial curvature is a quartic differential equation according to [6, 7, 8]. A global analysis of the lines of

axial curvature was developed in [6].

The mean directionally curved lines are globally defined and their singularities are either the inflection points, where the ellipse of curvature is a radial line segment, or the minimal points, where the mean curvature vector vanishes. It was shown in [11] that the differential equation of mean directionally curved lines fits into the class of quadratic or binary differential equations. The global behavior of mean directionally curved lines was studied in [11].

The asymptotic lines do not need to be globally defined on the surfaces and in general are not orthogonal. It was shown in [13] that a necessary and sufficient condition for existence of the globally defined asymptotic lines on a surface $\mathbf{M}^2$ in $\mathbf{R}^4$ is the local convexity of $\mathbf{M}^2$. The differential equation of asymptotic lines is also a quadratic differential equation and their singularities are the inflection points.

The ν-principal curvature lines are orthogonal and globally defined on surfaces immersed in $\mathbf{R}^4$ and their singularities are the ν-umbilic points, where the ν-principal curvatures coincide. The differential equation of ν-principal curvature lines is a quadratic differential equation according to [15]. An analysis of ν-principal curvature lines near generic ν-umbilic points is presented in [15] and in [5] the ν-principal cycles (closed ν-principal curvature lines) are studied. A global analysis of the ν-principal curvature lines was developed in [3], for $\nu = H$, where H is the normal mean curvature vector.

We prove in [12] that the orthogonality of the asymptotic lines is equivalent to the vanishing of the normal curvature. This result has been already obtained by Romero-Fuster and Sánchez-Bringas in [14] using a different approach. We also prove in [12] that the quartic differential equation of lines of axial curvature can be written as the product of the quadratic differential

equations of mean directionally curved lines and asymptotic lines if and only if the normal curvature of α vanishes at every point. Thus if the normal curvature of α vanishes at every point then the axial curvature cross fields split into four direction fields and therefore it is not possible that the lines of axial curvature have transversal crossings.

On the other hand, it is well known that a point p is semiumbilic if and only if the normal curvature vanishes at p, [14]. Semiumbilic points are interesting from the viewpoint of the theory of singularities of functions. Observe now that we have analogous statements if instead of vanishing normal curvature it is required semiumbilicity.

We say that an immersion $\alpha : \mathbf{M}^2 \to \mathbf{R}^4$ is *hyperspherical* if its image is contained in a hypersphere. In this work we study some properties of surfaces immersed in $\mathbf{R}^4$ whose asymptotic lines are orthogonal. In particular, we relate the property of having globally defined orthogonal asymptotic lines with hypersphericity, obtaining the following theorem.

Theorem 3.2. Let $\alpha : \mathbf{M}^2 \to \mathbf{R}^4$ be an immersion of a smooth oriented surface with globally defined orthogonal asymptotic lines. Suppose that there exist an unitary normal vector field ν and $r > 0$ such that the distance from the projection of the ellipse of curvature $\varepsilon_\alpha(p)$ onto the ν-axis to p is r, for all $p \in \mathbf{M}^2$, and the Gaussian curvature $K \neq r^2$. Then α is hyperspherical.

Finally, theorem 3.4 of [14], lemma 2.1 and theorem 2.1 of [12] and results of this paper are put together in Theorem 3.5 establishing seven other equivalent conditions to the orthogonality of the asymptotic lines.

This paper is organized as follows. A review of properties of the first and second fundamental forms, the ellipse of curvature and the line fields on surfaces immersed in $\mathbf{R}^4$ is presented in section 2. General aspects of the curvature theory for surfaces immersed in $\mathbf{R}^4$ are presented in the works of

Forsyth [2], Wong [17], Little [10] and Asperti [1]. Section 3 is devoted to the study of orthogonal asymptotic lines as well as hypersphericity of immersions. Finally, in section 4 some general problems are stated.

2 Line fields on surfaces in $\mathbf{R}^4$

For sake of completeness in this section we present a survey of the relevant notions that will need later. Let $\alpha : \mathbf{M}^2 \to \mathbf{R}^4$ be an immersion of a smooth oriented surface into $\mathbf{R}^4$, which is endowed with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and is oriented. In this paper immersions are assumed to be C^∞. Denote respectively by $\mathbf{TM}$ and $\mathbf{NM}$ the tangent and the normal bundles of α and by $T_p\mathbf{M}$ and $N_p\mathbf{M}$ the respective fibers, i.e., the tangent and the normal planes at $p \in \mathbf{M}^2$. Let $\{\nu_1, \nu_2\}$ be a frame of vector fields orthonormal to α. Assume that (u, v) is a positive chart of $\mathbf{M}^2$ and that $\{\alpha_u, \alpha_v, \nu_1, \nu_2\}$ is a positive frame of $\mathbf{R}^4$. In such a chart (u, v) the first fundamental form of α, I_α, is given by

$$I = I_\alpha = \langle d\alpha, d\alpha \rangle = Edu^2 + 2Fdudv + Gdv^2,$$

where $E = \langle \alpha_u, \alpha_u \rangle$, $F = \langle \alpha_u, \alpha_v \rangle$ and $G = \langle \alpha_v, \alpha_v \rangle$. The second fundamental form of α, II_α, is defined in terms of the $\mathbf{NM}$-valued quadratic form

$$II = II_\alpha = \langle d^2\alpha, \nu_1 \rangle \nu_1 + \langle d^2\alpha, \nu_2 \rangle \nu_2 = II_{\nu_1}\nu_1 + II_{\nu_2}\nu_2,$$

where

$$II_{\nu_i} = II_{\nu_i,\alpha} = e_i du^2 + 2f_i dudv + g_i dv^2,$$

$e_i = \langle \alpha_{uu}, \nu_i \rangle$, $f_i = \langle \alpha_{uv}, \nu_i \rangle$, and $g_i = \langle \alpha_{vv}, \nu_i \rangle$, for $i = 1, 2$.

The following functions are associated to α (see [10]):

1. The *mean curvature vector* of α

$$H = H_\alpha = H_1\nu_1 + H_2\nu_2,$$

where

$$H_i = H_{i,\alpha} = \frac{Eg_i - 2Ff_i + Ge_i}{2(EG - F^2)},$$

for $i = 1, 2$;

2. The *normal curvature* of α

$$k_N = k_{N,\alpha} = \frac{E(f_1g_2 - f_2g_1) - F(e_1g_2 - e_2g_1) + G(e_1f_2 - e_2f_1)}{2(EG - F^2)};$$

3. The *resultant* Δ of $II_{1,\alpha}$ and $II_{2,\alpha}$

$$\Delta = \Delta_\alpha = \frac{1}{4(EG - F^2)} \begin{vmatrix} e_1 & 2f_1 & g_1 & 0 \\ e_2 & 2f_2 & g_2 & 0 \\ 0 & e_1 & 2f_1 & g_1 \\ 0 & e_2 & 2f_2 & g_2 \end{vmatrix};$$

4. The *Gaussian curvature* of α

$$K = K_\alpha = \frac{e_1g_1 - (f_1)^2 + e_2g_2 - (f_2)^2}{EG - F^2};$$

5. The *normal curvature vector* of α defined by $\eta(p, v) = \frac{II(p,v)}{I(p,v)}$.

The image of the unitary tangent circle $\mathbf{S}^1$ by $\eta(p) : T_p\mathbf{M} \to N_p\mathbf{M}$ describes an ellipse in $N_p\mathbf{M}$ called *ellipse of curvature* of α at p and denoted by $\varepsilon_\alpha(p)$. This ellipse may degenerate into a line segment, a circle or a point. The center of the ellipse of curvature is the mean curvature vector H and the area of $\varepsilon_\alpha(p)$ is given by $\frac{\pi}{2}\,|k_N(p)|$. The map $\eta(p)$ restricted to $\mathbf{S}^1$, being quadratic, is a double covering of the ellipse of curvature. Thus every point of the ellipse corresponds to two diametrically opposed points of the unitary tangent circle. The ellipse of curvature is invariant by rotations in both the tangent and normal planes.

A point $p \in \mathbf{M}^2$ is called a *minimal point* of α if $H(p) = 0$ and it is called an *inflection point* of α if $\Delta(p) = 0$ and $k_N(p) = 0$. It follows that $p \in \mathbf{M}^2$

is an inflection point if and only if its ellipse of curvature is a radial line segment [10].

Lines of axial curvature. The four vertices of the ellipse of curvature $\varepsilon_\alpha(p)$ determine eight points on the unitary tangent circle which define two crosses in the tangent plane. Thus we have two cross fields on $\mathbf{M}^2$ called *axial curvature cross fields*. This construction fails at the *axiumbilic points* where the ellipse of curvature becomes either a circle or a point. Generically the index of an isolated axiumbilic point is $\pm\frac{1}{4}$ (see [6, 7, 8]). The integral curves of the axial curvature cross fields are the *lines of axial curvature*.

Generically there is no good way to distinguish one end of the large (or small) axis of $\varepsilon_\alpha(p)$ and therefore pick out a direction of the cross field. Thus a line of axial curvature is not necessarily a simple regular curve; it can be immersed with transversal crossings.

The differential equation of the lines of axial curvature is a quartic differential equation of the form

$$Jac\Big(\|\eta - H\|^2, I\Big) = 0, \tag{1}$$

where

$$Jac(\cdot,\cdot) = \frac{\partial(\cdot,\cdot)}{\partial(du,dv)},$$

which according to [6] can be written as

$$A_0du^4 + A_1du^3dv + A_2du^2dv^2 + A_3dudv^3 + A_4dv^4 = 0, \tag{2}$$

where

$$A_0 = a_0E^3,\ A_1 = a_1E^3,\ A_2 = -6a_0GE^2 + 3a_1FE^2,$$

$$A_3 = -8a_0EFG + a_1E(4F^2 - EG),\ A_4 = a_0G(EG - 4F^2) + a_1F(2F^2 - EG),$$

$$a_0 = 4\Big[F(EG - 2F^2)(e_1^2 + e_2^2) - Ea_6a_2 - E^2F(a_3 + a_5) + E^3a_4\Big],$$

$$a_1 = 4\Big[Ga_6(e_1^2+e_2^2) + 8EFGa_2 + E^3(g_1^2+g_2^2) - 2E^2G(a_3+a_5)\Big],$$

$$a_2 = e_1f_1 + e_2f_2,\ a_3 = e_1g_1 + e_2g_2,\ a_4 = f_1g_1 + f_2g_2,$$

$$a_5 = 2(f_1^2+f_2^2),\ a_6 = EG - 4F^2.$$

Mean directionally curved lines. The line through the mean curvature vector $H(p)$ meets $\varepsilon_\alpha(p)$ at two diametrically opposed points. This construction induces two orthogonal directions on $T_p\mathbf{M}^2$. Therefore we have two orthogonal direction fields on $\mathbf{M}^2$ called *H-direction fields*. The singularities of these fields, called here *H-singularities*, are the points where either $H = 0$ (minimal points) or at which the ellipse of curvature becomes a radial line segment (inflection points). Generically the index of an isolated H-singularity is $\pm\frac{1}{2}$ [11]. The integral curves of the H-direction fields are the *mean directionally curved lines*.

The differential equation of mean directionally curved lines is a quadratic differential equation of the form [11]

$$Jac\{Jac(II_{\nu_1}, II_{\nu_2}), I\} = 0, \tag{3}$$

which can be written as

$$B_1(u,v)du^2 + 2B_2(u,v)dudv + B_3(u,v)dv^2 = 0, \tag{4}$$

where

$$B_1 = (e_1g_2 - e_2g_1)E + 2(e_2f_1 - e_1f_2)F,\ B_2 = (f_1g_2 - f_2g_1)E + (e_2f_1 - e_1f_2)G,$$

$$B_3 = 2(f_1g_2 - f_2g_1)F + (e_2g_1 - e_1g_2)G.$$

Asymptotic lines. Suppose that p (the origin of $N_p\mathbf{M}^2$) lies outside $\varepsilon_\alpha(p)$, for all $p \in \mathbf{M}^2$. The two points on $\varepsilon_\alpha(p)$ at which the lines through the normal curvature vectors are tangent to $\varepsilon_\alpha(p)$ induce a pair of directions

in $T_p\mathbf{M}^2$ which in general are not orthogonal. Thus we have two tangent direction fields on $\mathbf{M}^2$, called *asymptotic direction fields*. The singularities of these fields are the points where the ellipse of curvature becomes a radial line segment, i.e., the *inflection points*. Generically the index of an isolated inflection point is $\pm\frac{1}{2}$ [4]. The integral curves of the asymptotic direction fields are the *asymptotic lines*.

The differential equation of asymptotic lines is a quadratic differential equation of the form [11]

$$Jac(II_{\nu_1}, II_{\nu_2}) = 0, \tag{5}$$

which can be written as

$$T_1(u,v)du^2 + T_2(u,v)dudv + T_3(u,v)dv^2 = 0, \tag{6}$$

where

$$T_1 = e_1 f_2 - e_2 f_1,\ T_2 = e_1 g_2 - e_2 g_1,\ T_3 = f_1 g_2 - f_2 g_1.$$

ν-Principal curvature lines. The projection of the pullback, $\alpha^*(\mathbf{R}^4)$, of the tangent bundle of $\mathbf{R}^4$ onto the tangent bundle of an immersion α will be denoted by $\Pi_{\alpha,T}$. This vector bundle is endowed with the standard metric induced by the Euclidean one in $\mathbf{R}^4$.

Denote by $\nu = \nu_\alpha$ the *unit normal vector field* of α. The eigenvalues $k_1 = k_{1,\alpha} \leq k_{2,\alpha} = k_2$ of the *Weingarten operator* $\mathcal{W}_\alpha = -\Pi_{\alpha,T} D\nu_\alpha$ of $\mathbf{TM}$ are called the *ν-principal curvatures* of α. The points where $k = k_1 = k_2$ will be called the *ν-umbilic* points of α and define the set $\mathcal{S}_u = \mathcal{S}_{u,\alpha}$. We say that α is *ν-umbilical* if each point of the immersion is ν-umbilic. Outside $\mathcal{S}_u$ are defined the *minimal*, $L_{m,\alpha}$, and the *maximal*, $L_{M,\alpha}$, *ν-principal line fields* of α, which are the eigenspaces of $\mathcal{W}_\alpha$ associated respectively to k_1 and k_2. Generically the index of an isolated ν-umbilic point is $\pm\frac{1}{2}$ [15]. The integral curves of the ν-principal line fields are the *ν-principal curvature lines*.

In a local chart (u, v) the ν-principal curvatures lines are characterized as the solutions of the following quadratic differential equation [15]

$$(Fg_\nu - f_\nu G)dv^2 + (Eg_\nu - e_\nu G)dudv + (Ef_\nu - Fe_\nu)du^2 = 0, \tag{7}$$

where E, F and G are the coefficients of the first fundamental form and $e_\nu =< \alpha_{uu}, \nu >$, $f_\nu =< \alpha_{uv}, \nu >$ and $g_\nu =< \alpha_{vv}, \nu >$ are the coefficients of the *second fundamental form relative to* ν, denoted by $II_\nu = II_{\nu_\alpha}$. Equation (7) is equivalently written as

$$Jac(II_\nu, I) = 0. \tag{8}$$

3 Orthogonal asymptotic lines

Let $\alpha : \mathbf{M}^2 \to \mathbf{R}^4$ be an immersion of a smooth oriented surface into $\mathbf{R}^4$. In [6] Garcia and Sotomayor prove the following theorem: Suppose that the image of the surface $\mathbf{M}^2$ by α is contained into $\mathbf{R}^3$. Then the quartic differential equation of lines of axial curvature is the product of the quadratic differential equation of its principal curvature lines and the quadratic differential equation of its mean curvature lines. It is interesting to observe that every point of $\mathbf{M}^2$ is an inflection point.

We have established in [11] the following theorem: Let $\alpha : \mathbf{M}^2 \to \mathbf{S}^3(r)$ be an immersion of a smooth oriented surface into a 3-dimensional sphere of radius $r > 0$. Consider the natural inclusion $i : \mathbf{S}^3(r) \to \mathbf{R}^4$ and the composition $i \circ \alpha$ also denoted by α. Then the quartic differential equation of lines of axial curvature (1) can be written as

$$Jac\{Jac(II_{\nu_1}, II_{\nu_2}), I\} \cdot Jac(II_{\nu_1}, II_{\nu_2}) = 0, \tag{9}$$

where the first expression in (9) is the quadratic differential equation of mean directionally curved lines (3) and the second one is the quadratic differential equation of asymptotic lines (5).

It is interesting to observe that in the above construction the asymptotic lines are orthogonal and the normal curvature of α vanishes at every point. This is a particular case of the following theorem proved in [12], which was also obtained in [14] using a different approach: Let $\alpha : \mathbf{M}^2 \to \mathbf{R}^4$ be an immersion of a smooth oriented surface with isolated inflection points. The immersion α has orthogonal asymptotic lines if and only if the normal curvature of α vanishes at every point.

We have established in [12] the following theorem: Let $\alpha : \mathbf{M}^2 \to \mathbf{R}^4$ be an immersion of a smooth oriented surface with isolated inflection points. The quartic differential equation of lines of axial curvature (1) can be written as

$$Jac\{Jac(II_{\nu_1}, II_{\nu_2}), I\} \cdot Jac(II_{\nu_1}, II_{\nu_2}) = 0, \tag{10}$$

where the first expression in (10) is the quadratic differential equation of mean directionally curved lines (3) and the second one is the quadratic differential equation of asymptotic lines (5), if and only if the normal curvature of α vanishes at every point.

We can prove the following corollary: Let $\alpha : \mathbf{M}^2 \to \mathbf{R}^4$ be an immersion of a smooth oriented surface into $\mathbf{R}^4$. If the immersion α has orthogonal asymptotic lines then the inflection points are obtained where the ellipse of curvature becomes a point. In fact, from Equation (10)

$$Jac\Big(\|\eta - H\|^2, I\Big) = Jac\{Jac(II_{\nu_1}, II_{\nu_2}), I\} \cdot Jac(II_{\nu_1}, II_{\nu_2}) = 0. \tag{11}$$

As the inflection points are singularities of asymptotic lines then by (11) they are singularities of lines of axial curvature. But the singularities of lines of axial curvature are the points where the ellipse of curvature becomes either a circle or a point. Thus the only possibility in this case is that the ellipse of curvature becomes a point.

Theorem 3.1 *Let $\alpha : \mathbf{M}^2 \to \mathbf{S}^3(r)$ be an immersion of a smooth oriented surface into a 3-dimensional sphere of radius $r > 0$. Consider the natural inclusion $i : \mathbf{S}^3(r) \to \mathbf{R}^4$ and the composition $i \circ \alpha$ also denoted by α. Then there exist an unitary normal vector field ν and $\lambda > 0$ such that the ellipse of curvature $\varepsilon_\alpha(p)$ is a line segment with the following property: the distance from the projection of $\varepsilon_\alpha(p)$ onto the ν-axis to p is λ, for all $p \in \mathbf{M}^2$.*

Proof. Let $\{\nu_1, \nu_2\}$ be a frame of vector fields orthonormal to α, where $\nu_1(p) \in T_p\mathbf{S}^3(r)$ and $\nu_2(p)$ is the inward normal to $\mathbf{S}^3(r)$, for all $p \in \mathbf{M}^2$. Thus

$$\nu_2 \equiv -\frac{1}{r}\,\alpha,\ e_2 = \frac{1}{r}\,E,\ f_2 = \frac{1}{r}\,F \text{ and } g_2 = \frac{1}{r}\,G,$$

where E, F and G are the coefficients of the first fundamental form of α. It follows that

$$II_{\nu_2} = \frac{1}{r}\,I.$$

Now

$$\eta = \frac{II}{I} = \frac{II_{\nu_1}}{I}\,\nu_1 + \frac{II_{\nu_2}}{I}\,\nu_2 = \frac{II_{\nu_1}}{I}\,\nu_1 + \frac{1}{r}\,\nu_2.$$

This implies that the ellipse of curvature $\varepsilon_\alpha(p)$ is a line segment orthogonal to ν_2, for all $p \in \mathbf{M}^2$. Define $\nu = \nu_2$ and $\lambda = \frac{1}{r}$. The theorem is proved.

Let $\alpha : \mathbf{M}^2 \to \mathbf{R}^4$ be an immersion of a smooth oriented surface with globally defined orthogonal asymptotic lines. Then the ellipse of curvature $\varepsilon_\alpha(p)$ is a line segment for all $p \in \mathbf{M}^2$ except at the inflection points. We say that the immersion α has *constant projection* if there exist an unitary normal vector field ν and $r > 0$ such that the distance from the projection of $\varepsilon_\alpha(p)$ onto the ν-axis to p (the origin of $N_p\mathbf{M}^2$) is r, for all $p \in \mathbf{M}^2$. The constant r is called *distance of projection*.

Theorem 3.1 shows that if α is hyperspherical then α has constant projection whose distance of projection is r^{-1}, where r is the radius of the hypersphere. The converse of Theorem 3.1 is given by the following theorem.

Theorem 3.2 *Let $\alpha : \mathbf{M}^2 \to \mathbf{R}^4$ be an immersion of a smooth oriented surface with globally defined orthogonal asymptotic lines. Suppose that α has constant projection with distance of projection $r > 0$, and the Gaussian curvature $K \neq r^2$. Then α is hyperspherical.*

Proof. Since all the notions of this paper are independents of the chart it is enough to prove this theorem for an orthogonal one. By hypothesis there is an unitary normal vector field ν orthogonal to $\varepsilon_\alpha(p)$, for all $p \in \mathbf{M}^2$. We can take $\{\nu_1 = \nu^\perp, \nu_2 = \nu\}$ a frame of vector fields orthonormal to α, where $\nu^\perp(p)$ is parallel to $\varepsilon_\alpha(p)$, such that $\{\alpha_u, \alpha_v, \nu^\perp, \nu\}$ is a positive frame of $\mathbf{R}^4$, for a positive orthogonal chart (u, v) of $\mathbf{M}^2$. Thus $e_2 = rE,\ f_2 = 0,\ g_2 = rG$. The immersion α satisfies the Codazzi equations [2]

$$(e_1)_v - (f_1)_u = \Gamma^1_{12} e_1 + \left(\Gamma^2_{12} - \Gamma^1_{11}\right) f_1 - \Gamma^2_{11} g_1 - a^3_{12} e_2 + a^3_{11} f_2, \tag{12}$$

$$(e_2)_v - (f_2)_u = \Gamma^1_{12} e_2 + \left(\Gamma^2_{12} - \Gamma^1_{11}\right) f_2 - \Gamma^2_{11} g_2 - a^3_{12} e_1 + a^3_{11} f_1, \tag{13}$$

$$(f_1)_v - (g_1)_u = \Gamma^1_{22} e_1 + \left(\Gamma^2_{22} - \Gamma^1_{12}\right) f_1 - \Gamma^2_{12} g_1 + a^3_{12} f_2 - a^3_{11} g_2, \tag{14}$$

$$(f_2)_v - (g_2)_u = \Gamma^1_{22} e_2 + \left(\Gamma^2_{22} - \Gamma^1_{12}\right) f_2 - \Gamma^2_{12} g_2 - a^3_{12} f_1 + a^3_{11} g_1, \tag{15}$$

and the following structure equations [2]

$$(\nu^\perp)_u = a^1_{11}\alpha_u + a^2_{11}\alpha_v + a^3_{11}\nu, \tag{16}$$

$$(\nu^\perp)_v = a^1_{12}\alpha_u + a^2_{12}\alpha_v + a^3_{12}\nu, \tag{17}$$

$$\nu_u = a^1_{21}\alpha_u + a^2_{21}\alpha_v - a^3_{11}\nu^\perp, \tag{18}$$

$$\nu_v = a^1_{22}\alpha_u + a^2_{22}\alpha_v - a^3_{12}\nu^\perp, \tag{19}$$

where

$$a^1_{11} = \frac{f_1 F - e_1 G}{EG - F^2},\ a^2_{11} = \frac{e_1 F - f_1 E}{EG - F^2},\ a^1_{12} = \frac{g_1 F - f_1 G}{EG - F^2},\ a^2_{12} = \frac{f_1 F - g_1 E}{EG - F^2},$$

$$a_{21}^1=\frac{f_2F-e_2G}{EG-F^2},\ a_{21}^2=\frac{e_2F-f_2E}{EG-F^2},\ a_{22}^1=\frac{g_2F-f_2G}{EG-F^2},\ a_{22}^2=\frac{f_2F-g_2E}{EG-F^2},$$

and Γ_{ij}^k are the Christoffel symbols of α [2], $i,j,k=1,2$, which in this case are given by

$$\Gamma_{11}^1=\frac{E_u}{2E},\ \Gamma_{11}^2=-\frac{E_v}{2G},\ \Gamma_{12}^1=\frac{E_v}{2E},\ \Gamma_{12}^2=\frac{G_u}{2G},\ \Gamma_{22}^1=-\frac{G_u}{2E},\ \Gamma_{22}^2=\frac{G_v}{2G}.$$

Substituting the above Christoffel symbols in the Codazzi equations (13) and (15) we have respectively

$$rE_v=\frac{E_v}{2E}rE+\frac{E_v}{2G}rG-a_{12}^3e_1+a_{11}^3f_1 \tag{20}$$

and

$$-rG_u=-\frac{G_u}{2E}rE-\frac{G_u}{2G}rG-a_{12}^3f_1+a_{11}^3g_1. \tag{21}$$

But Equations (20) and (21) are equivalents to

$$-a_{12}^3e_1+a_{11}^3f_1=0 \tag{22}$$

and

$$-a_{12}^3f_1+a_{11}^3g_1=0, \tag{23}$$

respectively. Now the Gaussian curvature is

$$K=\frac{e_1g_1-(f_1)^2}{EG}+\frac{e_2g_2}{EG}=\frac{e_1g_1-(f_1)^2}{EG}+r^2.$$

By hypothesis $K\neq r^2$, and thus

$$e_1g_1-(f_1)^2\neq 0. \tag{24}$$

From the Equations (22), (23) and (24) we have that

$$a_{11}^3=a_{12}^3=0. \tag{25}$$

Substituting Equation (25) in (18) and (19) results that

$$\nu_u = -r\alpha_u \text{ and } \nu_v = -r\alpha_v.$$

Thus

$$\nu = -r\alpha + \gamma,$$

where γ is a constant vector. Therefore

$$\alpha = \frac{\gamma}{r} - \frac{1}{r}\nu.$$

This means that $\alpha(\mathbf{M}^2)$ belongs to a hypersphere with center $\frac{\gamma}{r}$ and radius $\frac{1}{r}$. The theorem is proved.

The proof of the following theorem is immediate from the proof of Theorem 3.1.

Theorem 3.3 *Let $\alpha : \mathbf{M}^2 \to \mathbf{S}^3(r)$ be an immersion of a smooth oriented surface into a 3-dimensional sphere of radius $r > 0$. Consider the natural inclusion $i : \mathbf{S}^3(r) \to \mathbf{R}^4$ and the composition $i \circ \alpha$ also denoted by α. Then there exist an unitary normal vector field ν and $\lambda > 0$ such that $II_\nu = \langle d^2\alpha, \nu\rangle = \lambda I$.*

The converse of Theorem 3.3 is given by the following theorem.

Theorem 3.4 *Let $\alpha : \mathbf{M}^2 \to \mathbf{R}^4$ be an immersion of a smooth oriented surface. Suppose that ν is an unitary normal vector field such that $II_\nu = \langle d^2\alpha, \nu\rangle = \lambda I$, where λ is a nonzero constant, and the Gaussian curvature $K \neq \lambda^2$. Then α is hyperspherical.*

Proof. Take the positive frame $\{\alpha_u, \alpha_v, \nu^\perp, \nu\}$. As $II_\nu = \langle d^2\alpha, \nu\rangle = \lambda I$ we have

$$\eta = \frac{II}{I} = \frac{II_{\nu^\perp}}{I}\,\nu^\perp + \frac{II_\nu}{I}\,\nu = \frac{II_{\nu^\perp}}{I}\,\nu^\perp + \lambda\,\nu.$$

This implies that the ellipse of curvature $\varepsilon_\alpha(p)$ is a line segment whose distance from their projection onto the ν-axis to p is constant and equal to λ, for all $p \in \mathbf{M}^2$. Therefore α has constant projection with distance of projection $\lambda > 0$. As $K \neq \lambda^2$ the theorem follows from Theorem 3.2.

Let $\alpha : \mathbf{M}^2 \to \mathbf{R}^4$ be an immersion of a smooth oriented surface with globally defined orthogonal asymptotic lines. Then the normal curvature of α vanishes at every point. So there exist normal vector fields ν and $\nu^\perp$ such that

$$\eta = \frac{II}{I} = \frac{II_{\nu^\perp}}{I}\,\nu^\perp + \frac{II_\nu}{I}\,\nu = \frac{II_{\nu^\perp}}{I}\,\nu^\perp + \lambda\,\nu.$$

Thus $II_\nu = \lambda I$, where λ is a positive scalar function on $\mathbf{M}^2$. This implies that α is ν-umbilical. The differential equation of asymptotic lines (5) is given by

$$0 = Jac(II_{\nu^\perp}, II_\nu) = Jac(II_{\nu^\perp}, \lambda I),$$

which is equivalent to

$$Jac(II_{\nu^\perp}, I) = 0.$$

But this equation is the differential equation of $\nu^\perp$-principal curvature lines (8).

Theorem 3.4 of [14], lemma 2.1 and theorem 2.1 of [12] and above results are put together in the next theorem.

Theorem 3.5 *Let $\alpha : \mathbf{M}^2 \to \mathbf{R}^4$ be an immersion of a smooth oriented surface. The following are equivalent conditions on α:*

a) The immersion α has everywhere defined orthogonal asymptotic lines;

b) The normal curvature of α vanishes at every point;

c) The immersion α is ν-umbilical for some unitary normal vector field ν;

d) All points of α are semiumbilic;

e) There exist a positive scalar function λ and an unitary normal vector field ν such that the second fundamental form relative to ν is given by $II_\nu = \lambda I$;

f) The asymptotic lines coincide with the lines of axial curvature defined by the large axis of the ellipse of curvature;

g) The asymptotic lines coincide with the $\nu^\perp$-principal curvature lines, for some unitary normal vector field ν;

h) The quartic differential equation of lines of axial curvature is the product of the quadratic differential equations of mean directionally curved lines and asymptotic lines.

Furthermore if the above function λ is a nonzero constant and the Gaussian curvature $K \neq \lambda^2$ then α is hyperspherical.

4 Concluding remarks

One direction of research can be stated: To give an example of a non-hyperspherical immersion α of a smooth oriented surface in $\mathbf{R}^4$ with globally defined orthogonal asymptotic lines having an isolated inflection point.

Other direction of research emerges with the evaluation of the index of an isolated ν-umbilic point. This is related to the upper bound 1 for the umbilic index on surfaces immersed in $\mathbf{R}^3$ and the Carathéodory conjecture (see [16] and references therein). Gutierrez and Sánchez-Bringas [9] have shown that this bound does not hold for the ν approach.

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